ON THE ALTERNATING PROJECTIONS THEOREM AND BIVARIATE STATIONARY STOCHASTIC PROCESSES

BY

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Summary. In this paper we shall first use the theorem of von Neumann on alternating projections to obtain an algorithm for finding the projection of an element x in a Hilbert space $\mathscr{H}$ onto the subspace spanned by $\mathscr{H}$-valued orthogonally scattered measures ξ_1 and ξ_2. We then specialize this algorithm to the case that ξ_1 and ξ_2 are the canonical measures of the components of a bivariate stationary stochastic process (SP), and thereby get an algorithm for finding the best linear predictor in the time domain.

1. **Preliminary results.** In this section we state some results which are used in later sections.

The following theorem is due to J. von Neumann (cf. [7, p. 55]).

THEOREM 1.1 (ALTERNATING PROJECTIONS). *Let P_1, P_2, and T be projection operators on a Hilbert space $\mathscr{H}$ onto the subspaces $\mathscr{M}_1$, $\mathscr{M}_2$, and $\mathscr{M}_1 \cap \mathscr{M}_2$. If T_n is the nth term of either of the sequences*

$$P_1, P_2P_1, P_1P_2P_1, P_2P_1P_2P_1, \ldots,$$
$$P_2, P_1P_2, P_2P_1P_2, P_1P_2P_1P_2, \ldots,$$

then $T_n \to T$ strongly(2), *as $n \to \infty$.*

This at once yields the following corollary, which we need later.

COROLLARY 1.2. *With the notation of* 1.1, *if P is the projection operator on $\mathscr{H}$ onto the subspace $\mathfrak{S}(\mathscr{M}_1+\mathscr{M}_2)$ which is spanned by $\mathscr{M}_1$ and $\mathscr{M}_2$, then*

$$P = P_1+P_2-P_1P_2-P_2P_1+P_1P_2P_1+P_2P_1P_2-\cdots$$

the convergence being in the strong sense.

The next theorem is due to A. S. Besicovitch (cf. [1, p. 9]).

THEOREM 1.3. *Let* (i) *μ be a bounded, countably additive* (*c.a.*), *nonnegative measure defined on the family $\mathscr{B}$ of Borel subsets of the real line R.*

(ii) *ν be a c.a., complex-valued measure on $\mathscr{B}$.*

(iii) *$D(\lambda, h)=\{\nu(\lambda-h, \lambda+h)/\mu(\lambda-h, \lambda+h)\}\chi_{\sigma(\mu)}(\lambda)$, where $\sigma(\mu)$ is the spectrum of μ, i.e., $\sigma(\mu)=\{\lambda$: for each $h>0$, $\mu(\lambda-h, \lambda+h)>0\}$. Then* (a) *the limit $D(\lambda, h)$, as $h \to 0$, exists (finite) almost everywhere with respect to μ (a.e. μ).*

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(2) I.e., for each x in $\mathscr{H}$, $|T_nx-Px| \to 0$, as $n \to \infty$, $|\quad|$ being the norm in $\mathscr{H}$.

(b) *If ν is absolutely continuous* (*a.c.*) *with respect to μ and if the Radon-Nikodym derivative of ν with respect to μ is $d\nu/d\mu$, then*

$$(d\nu/d\mu)(\cdot) = \lim D(\cdot, h), \text{ as } h \to 0, \text{ a.e. } \mu.$$

We shall denote $\lim D(\cdot, h)$, as $h \to 0$, by $(D\nu/D\mu)(\cdot)$ and shall call it the *Besicovitch derivative* of ν with respect to μ.

The following is an immediate consequence of 1.3.

COROLLARY 1.4. *Let* (i) M_i $(i=1, 2)$ *be a σ-finite, c.a., nonnegative measure on the family $\mathscr{B}$ of Borel subsets of the real line R.*

(ii) $\mathscr{B}_i=\{B\colon B\in\mathscr{B}_i$ *and* $M_i(B)<\infty\}$, M_{12} *the product measure generated by* M_1 *and* M_2, *and* $\mathscr{B}_{12}=\{B\colon B$ *is a Borel subset of the complex plane* R^2 *and* $M_{12}(B)<\infty\}$.

(iii) μ *be a c.a. measure on the ring* $\mathscr{B}_{12}$ *which is a.c. with respect to* M_{12}.

(iv) $d\mu/dM_{12}$ *be any version of the Radon-Nikodym derivative of μ with respect to* M_{12}, *which is defined throughout* R^2.

(v) *For each* $B\in\mathscr{B}_1$ *and each* $t\in R$,

$$p(B, t) = \int_B (d\mu/dM_{12})(s, t)M_1(ds).$$

Then (a) *for each* $B\in\mathscr{B}_1$, $\mu_B(\cdot)=\mu(B\times\cdot)$ *is a c.a. measure on* $\mathscr{B}_2$ *which is a.c. with respect to* M_2.

(b) *For each* $B\in\mathscr{B}_1$, $p(B, \cdot)$ *is a version of* $d\mu_B/dM_2$, *the domain of which is* R.

(c) *For each* $t\in R$, $p(\cdot, t)$ *is a c.a. measure on* $\mathscr{B}_1$, *which is a.c. with respect to* M_1.

(d) *Analogous results to* (a), (b), *and* (c) *hold for* ν_B, *where for each* $B\in\mathscr{B}_2$, $\nu_B(\cdot)=\mu(\cdot\times B)$.

2. **Orthogonally scattered, $\mathscr{H}$-valued measures.** We first state a few results concerning measures whose values are orthogonally scattered in a given Hilbert space. These are special versions of more general results contained in [9], [10], and [11].

DEFINITION 2.1. Let (i) M be a σ-finite, c.a., nonnegative measure on the family $\mathscr{B}$ of Borel subsets of the real line R.

(ii) $\mathscr{B}_0=\{B\colon B\in\mathscr{B}$ and $M(B)<\infty\}$.

Then a function ξ on $\mathscr{B}_0$ into a given Hilbert space $\mathscr{H}$ such that for each pair of $B, C\in\mathscr{B}_0$,

$$(\xi(B), \xi(C)) = M(B\cap C)$$

is called an *orthogonally scattered* (*o.s.*) *$\mathscr{H}$-valued* measure over $(R, \mathscr{B})$. M is called the *spectral measure* associated with ξ. It follows immediately that $(\xi(B), \xi(C))=0$ if B and C are disjoint sets in $\mathscr{B}_0$, and $\xi(\bigcup_k B_k)=\sum_k \xi(B_k)$ if B_k's are disjoint sets in $\mathscr{B}_0$ whose union $\bigcup_k B_k$ is in $\mathscr{B}_0$.

There is a well-established theory of integration with respect to an o.s. $\mathscr{H}$-valued measure ξ whose associated spectral measure is σ-finite (cf. [10] and [11]). We will briefly state the main result.

THEOREM 2.2. *If* $\mathscr{S}$ *is the set of all stochastic integrals* $\int_R \phi \, d\xi$ *then*

(a) $\int_R \phi \, d\xi$ *exists iff* $\int_R |\phi|^2 \, dM < \infty$.

(b) $\int_R (a\phi + b\psi) \, d\xi = a \int_R \phi \, d\xi + b \int_R \psi \, d\xi$, *where* a *and* b *are complex numbers.*

(c) $\mathscr{S}$ = *the cyclic subspace spanned by* ξ, *i.e.*,

$$\mathscr{S} = \mathfrak{S}\{\xi(B); B \in \mathscr{B}_0\}.$$

(d) $(\int_R \phi \, d\xi, \int_R \psi \, d\xi) = \int_R \phi \bar{\psi} \, dM$.

(e) *The correspondence* $\phi \to \int_R \phi \, d\xi$ *is an isomorphism from* $L_2(R, \mathscr{B}, M)$ *onto* $\mathscr{S}$ *such that* (a) *and* (b) *hold, and therefore*

$$\left| \int_R \phi \, d\xi \right|^2 = \int_R |\phi|^2 \, dM.$$

Let x be in $\mathscr{H}$ and $\{\xi(m)\}$ be an orthonormal basis for a subspace $\mathscr{M}$ of $\mathscr{H}$. Then $\sum_m |(x, \xi(m))|^2 < \infty$ and the orthogonal projection of x onto $\mathscr{M}$ is given by $\sum_m (x, \xi(m))\xi(m)$. The following theorem generalizes this result.

THEOREM 2.3. *Let* (i) $x \in \mathscr{H}$ *and* ξ *be an o.s.* $\mathscr{H}$*-valued measure over* $(R, \mathscr{B})$ *whose associated spectral measure* M *is* σ*-finite.*

(ii) P *be the projection operator on* $\mathscr{H}$ *onto the cyclic subspace* $\mathscr{M}$ *spanned by the measure* ξ, *i.e.*, $\mathscr{M} = \mathfrak{S}\{\xi(B) : B \in \mathscr{B}_0\}$, *where* $\mathscr{B}_0 = \{B \in \mathscr{B} \;\&\; M(B) < \infty\}$.

Then (a) *the complex-valued functions* (x, ξ) *and* (ξ, x) *defined by* $(x, \xi)(\cdot) = (x, \xi(\cdot))$ *and* $(\xi, x)(\cdot) = (\xi(\cdot), x)$ *are c.a. measures over* $(R, \mathscr{B})$ *which are a.c. with respect to* M.

(b) $(x, \xi) = (\xi, x)^*$ *and* $d(x, \xi)/dM = \{d(\xi, x)/dM\}^*$.

(c) $d(x, \xi)/dM$ *and* $d(\xi, x)/dM \in L_2(R, \mathscr{B}, M)$, *and*

$$Px = \int_R \{d(x, \xi)/dM\} \, d\xi.$$

Proof. Since the proof of (a) and (b) is trivial, we proceed to give the proof of (c). By 2.2(c), there exists a complex-valued function ϕ such that

$$Px = \int_R \phi \, d\xi \quad \& \quad \phi \in L_2(R, \mathscr{B}, M). \tag{1}$$

Now by (1) and 2.2(d), for each $B \in \mathscr{B}_0$

$$\begin{aligned}(x, \xi)(B) &= (x, \xi(B)) = (Px, \xi(B)) \\ &= \left(\int_R \phi \, d\xi, \int_R \chi_B \, d\xi \right) = \int_B \phi \, dM.\end{aligned}$$

It immediately follows from the Besicovitch Theorem 1.3 that

$$d(x, \xi)/dM = \phi \text{ a.e. } M. \tag{2}$$

By (b), (1), and (2) the result follows. (Q.E.D.)

COROLLARY 2.4. *Let* (i) ξ *be an o.s.* $\mathscr{H}$*-valued measure over* $(R, \mathscr{B})$ *whose associated spectral measure* M *is* σ*-finite.*

(ii) $\phi \in L_2(R, \mathscr{B}, M)$.

Then (a) *for each* $x \in \mathscr{H}$, *the complex-valued function* (ξ, x) *defined in* 2.3 *is a c.a. measure over* $(R, \mathscr{B})$ *which is a.c. with respect to* M.

(b) $\phi \in L_1[R, \mathscr{B}, (\xi, x)]$, *and*

$$\left(\int_R \phi(\omega)\xi(d\omega), x\right) = \int_R \phi(\omega)(\xi, x)(d\omega).$$

Proof. The proof of (a) follows from 2.3(a). Let P denote the projection operator on $\mathscr{H}$ onto the cyclic subspace $\mathscr{M}$ spanned by the measure ξ. Since $\phi \in L_2(R, \mathscr{B}, M)$ and by 2.3(c), $d(\xi, x)/dM \in L_2(R, \mathscr{B}, M)$, therefore $\phi\; d(x, \xi)/dM \in L_1(R, \mathscr{B}, M)$ or equivalently $\phi \in L_1[R, \mathscr{B}, (\xi, x)]$. To prove the second part of (c), we observe that by 2.2(c), $\int_R \phi\, d\xi \in \mathscr{M}$ and therefore

$$\left(\int_R \phi\, d\xi, x\right) = \left(\int_R \phi\, d\xi, Px\right). \tag{1}$$

Now by 2.2(d), 2.3(b), and 2.3(c),

$$\begin{aligned}\left(\int_R \phi\, d\xi, Px\right) &= \left(\int_R \phi\, d\xi, \int_R \frac{d(x, \xi)}{dM}\, d\xi\right) = \int_R \phi(\omega) \frac{(d(x, \xi))^*}{dM} (\omega) M(d\omega) \\ &= \int_R \phi(\omega) \frac{d(\xi, x)}{dM} (\omega) M(d\omega) = \int_R \phi(\omega)(\xi, x)(d\omega).\end{aligned} \tag{2}$$

By (1) and (2) we get the result. (Q.E.D.)

3. **The alternating projections theorem and** $\mathscr{H}$**-valued measures.** Let $\{\xi_1(m)\}$ and $\{\xi_2(m)\}$ be orthonormal bases for the subspaces $\mathscr{M}_1$ and $\mathscr{M}_2$ of a given Hilbert space $\mathscr{H}$. Let P_1, P_2, and P be projection operators on $\mathscr{H}$ onto the subspaces $\mathscr{M}_1$, $\mathscr{M}_2$, and $\mathfrak{S}(\mathscr{M}_1 + \mathscr{M}_2)$. From 1.2 it follows that for each x in $\mathscr{H}$

$$\begin{aligned} Px &= \sum_m (x, \xi_1(m))\xi_1(m) + \sum_m (x, \xi_2(m))\xi_2(m) \\ &\quad - \sum_n \left\{\sum_m (x, \xi_2(m))(\xi_2(m), \xi_1(n))\right\}\xi_1(n) \\ &\quad - \sum_n \left\{\sum_m (x, \xi_1(m))(\xi_1(m), \xi_2(n))\right\}\xi_2(n) \\ &\quad + \cdots. \end{aligned} \tag{A}$$

Now an orthonormal basis is just an o.s. $\mathscr{H}$-valued measure over $(R, \mathscr{B})$ which is concentrated on the set of integers. In many situations it is more natural to think of a subspace $\mathscr{M}$ as being spanned by such a measure than by some basis, and to represent a vector in $\mathscr{H}$ by an integral rather than by a sum. (This happens, for instance, in the theory of the Fourier integral.)

We shall generalize equation (A) to the case in which $\mathscr{M}_1$ and $\mathscr{M}_2$ are spanned by such measures ξ_1 and ξ_2. We shall show that under certain conditions we get instead of (A):

$$\begin{aligned} Px = {} & \int_R \frac{d(x, \xi_1)}{dM_1}(s)\xi_1(ds) + \int_R \frac{d(x, \xi_2)}{dM_2}(s)\xi_2(ds) \\ & - \int_R \left\{ \int_R \frac{d(x, \xi_2)}{dM_2}(s) \frac{d(\xi_2, \xi_1)}{dM_{21}}(s, t) M_2(ds) \right\} \xi_1(dt) \\ & - \int_R \left\{ \int_R \frac{d(x, \xi_1)}{dM_1}(s) \frac{d(\xi_1, \xi_2)}{dM_{12}}(s, t) M_1(ds) \right\} \xi_2(dt) + \cdots \end{aligned} \tag{B}$$

where M_{12}, M_{21} are the product measures $M_1 \times M_2$, $M_2 \times M_1$, and $d(\ )/d$ denotes the Radon-Nikodym derivative.

NOTATION 3.1. $\mathscr{B}$ will denote the family of Borel subsets of the real line R. $\mathscr{H}$ will denote a fixed Hilbert space. ξ_i $(i=1, 2)$ will denote an o.s. $\mathscr{H}$-valued measure over $(R, \mathscr{B})$ whose associated spectral measure M_i $(i=1, 2)$ is σ-finite. We set

$\mathscr{B}_i = \{B : B \in \mathscr{B}$ & $M_i(B) < \infty\}$, $1 \leqq i \leqq 2$,

$\mathscr{B} \times \mathscr{B}$ = the family of Borel subsets of the complex plane R^2.

$\mathscr{B}_i \times \mathscr{B}_j = \{B \times C : B \in \mathscr{B}_i$ & $C \in \mathscr{B}_j\}$, $1 \leqq i, j \leqq 2$,

$\mathscr{R}_{ij}$ = the ring generated by $\mathscr{B}_i \times \mathscr{B}_j$, $1 \leqq i, j \leqq 2$,

M_{ij} = the product measure $M_i \times M_j$ on $\mathscr{B} \times \mathscr{B}$, $1 \leqq i, j \leqq 2$,

$\mathscr{B}_{ij} = \{B : B \in \mathscr{B} \times \mathscr{B}$ & $M_{ij}(B) < \infty\}$, $1 \leqq i, j \leqq 2$.

To be able to carry out our work we need to study the complex-valued measure (ξ_i, ξ_j) which is generated by ξ_i and ξ_j. Our definition of this measure will be given in several steps.

DEFINITION 3.2. We define (ξ_i, ξ_j) on $\mathscr{B}_i \times \mathscr{B}_j$ by:

$$(\xi_i, \xi_j)(B \times C) = (\xi_i(B), \xi_j(C)).$$

A usual argument may be used to prove the following lemma.

LEMMA 3.2. *Let* $\{B_k\}_{k=1}^m$ *and* $\{C_l\}_{l=1}^n$ *be finite sequences of sets in* $\mathscr{B}_i \times \mathscr{B}_j$ *such that* $\bigcup_{k=1}^m B_k = \bigcup_{l=1}^n C_l$. *Then*

$$\sum_{k=1}^m (\xi_i, \xi_j)(B_k) = \sum_{l=1}^n (\xi_i, \xi_j)(C_l).$$

Since every $B \in \mathscr{R}_{ij}$ is a finite, disjoint union of sets in $\mathscr{B}_i \times \mathscr{B}_j$ (cf. [3, p. 139]), from 3.2 it follows that (ξ_i, ξ_j) can be defined on $\mathscr{R}_{ij}$ in the following way.

DEFINITION 3.3. Let $B \in \mathscr{R}_{ij}$. Then we define

$$(\xi_i, \xi_j)(B) = \sum_{k=1}^n (\xi_i, \xi_j)(B_k),$$

where the B_k's are any finite sequence of disjoint sets in $\mathscr{B}_i \times \mathscr{B}_j$ such that $B = \bigcup_{k=1}^n B_k$.

The next lemma whose proof is easily seen shows that (ξ_i, ξ_j) is finitely additive on $\mathscr{R}_{ij}$.

LEMMA 3.4. (a) *(ξ_i, ξ_j) is finitely additive on $\mathscr{R}_{ij}$ and is a.c. with respect to M_{ij} on $\mathscr{R}_{ij}$.*

(b) *(ξ_i, ξ_j) is the unique finitely additive extension of the measure defined in* 3.2 *on $\mathscr{B}_i \times \mathscr{B}_j$ to $\mathscr{R}_{ij}$.*

To proceed further, some restriction has to be imposed on ξ_1 and ξ_2.

ASSUMPTION 3.5. There exist functions φ_{12} and φ_{21} on R^2 such that

(1) $\varphi_{ij}(\cdot, t) \in L_2(R, \mathscr{B}, M_i)$ a.e. $t(M_j)$,
$M_j-\text{ess. lub.}_{t\in R} |\varphi_{ij}(\cdot, t)|_{2,M_i} < \infty$(3).

(2) For each $B \in \mathscr{B}_{ij}$, $\chi_B\varphi_{ij} \in L_1(R^2, \mathscr{B}\times\mathscr{B}, M_{ij})$.

(3) For each $B \times C \in \mathscr{B}_i \times \mathscr{B}_j$

$$(\xi_i(B), \xi_j(C)) = \iint_{B\times C} \varphi_{ij}(s, t) M_{ij}(d(s, t)).$$

We will see later (cf. 4.5) that in many situations Assumption 3.5 is satisfied.

LEMMA 3.6. *Let* (i) *ξ_1 and ξ_2 be o.s. $\mathscr{H}$-valued measures over $(R, \mathscr{B})$ satisfying Assumption* 3.5. *Then* (a) *the measure (ξ_i, ξ_j) on R_{ij}, introduced in* 3.3, *has a c.a. extension μ_{ij} to $\mathscr{B}_{ij}$ such that for each $B \in \mathscr{B}_{ij}$,*

$$\mu_{ij}(B) = \iint_B \varphi_{ij}(s, t) M_{ij}(d(s, t)).$$

(b) *μ_{ij} is a.c. with respect to M_{ij} and $d\mu_{ij}/dM_{ij} = \varphi_{ij}$ a.e. M_{ij}.*

DEFINITION 3.7. We define (ξ_i, ξ_j) on $\mathscr{B}_{ij}$ to be the μ_{ij} of the last lemma.

In the next theorem we state some consequences which occur when we have a c.a. measure (ξ_i, ξ_j) as above. The proof is immediate from 1.4.

THEOREM 3.8. *With the notation of* 3.1, *if (ξ_i, ξ_j) is any c.a. measure on $\mathscr{B}_{ij}$ which is a.c. with respect to M_{ij} such that for each $B \times C \in \mathscr{B}_i \times \mathscr{B}_j$, $(\xi_i, \xi_j)(B \times C) = (\xi_i(B), \xi_j(C))$. Then* (a) *for each $B \in \mathscr{B}_j$, $(\xi_i, \xi_j(B))$ is a c.a. measure on $\mathscr{B}_i$ which is a.c. with respect to M_i.*

(b) *For each $B \in \mathscr{B}_j$, $\int_B \{d(\xi_i, \xi_j)/dM_{ij}\}(\cdot, t) M_j(dt)$ is a version of $d(\xi_i, \xi_j(B))/dM_i$.*

(c) *For almost all $s(M_i)$, $\{d(\xi_i, \xi_j(\cdot))/dM_i\}(s)$ is a c.a. measure on $\mathscr{B}_j$ which is a.c. with respect to M_j.*

(d) *For each $B \in \mathscr{B}_i$ similar results hold for the measure $(\xi_i(B), \xi_j)$.*

We shall now state the main theorem of this section.

(3) By definition $|\varphi_{ij}(\cdot, t)|_{2,M_i} = \left[\int_R |\varphi_{ij}(s, t)|^2 M_i(ds)\right]^{1/2}$.

THEOREM 3.9. *With the notation of* 3.1, *if* ξ_i $(i=1,2)$ *are o.s.* $\mathscr{H}$*-valued measures over* $(R,\mathscr{B})$ *satisfying Assumption* 3.5, *and if* (ξ_i,ξ_j) *is the c.a. measure on* $\mathscr{B}_{ij}$ *given by* 3.7, *then for each* x *in* $\mathscr{H}$

(a)
$$d(x,\xi_i)/dM_i \in L_2(R,\mathscr{B},M_i),$$
$$\frac{d(x,\xi_i)}{dM_i}(\cdot)\frac{d(\xi_i,\xi_j)}{dM_{ij}}(\cdot,t) \in L_1(R,\mathscr{B},M_i) \text{ a.e. } t(M_j),$$
$$\int_R \frac{d(x,\xi_i)}{dM_i}(s)\frac{d(\xi_i,\xi_j)}{dM_{ij}}(s,\cdot)M_i(ds) \in L_2(R,\mathscr{B},M_j),$$
$$\left\{\int_R \frac{d(x,\xi_i)}{dM_i}(s)\frac{d(\xi_i,\xi_j)}{dM_{ij}}(s,\cdot)M_i(ds)\right\}\frac{d(\xi_j,\xi_i)}{dM_{ji}}(\cdot,u) \in L_1(R,\mathscr{B},M_j) \text{ a.e. } u(M_i),$$
$$\int_R\left\{\int_R \frac{d(x,\xi_i)}{dM_i}(s)\frac{d(\xi_i,\xi_j)}{dM_{ij}}(s,t)M_i(ds)\right\}\frac{d(\xi_j,\xi_i)}{dM_{ji}}(t,\cdot)M_j(dt) \in L_2(R,\mathscr{B},M_i),\ldots.$$

(b) *If* P_i $(i=1,2)$ *is the projection operator onto the cyclic subspaces* $\mathscr{M}_i$ $(i=1,2)$ *which is spanned by* ξ_i $(i=1,2)$, *then*

$$P_ix = \int_R \frac{d(x,\xi_i)}{dM_i}(s)\xi_i(ds),$$
$$P_jP_ix = \int_R\left\{\int_R \frac{d(x,\xi_i)}{dM_i}(s)\frac{d(\xi_i,\xi_j)}{dM_{ij}}(s,t)M_i(ds)\right\}\xi_j(dt),$$
$$P_iP_jP_ix = \int_R\left\{\int_R\left\{\int_R \frac{d(x,\xi_i)}{dM_i}(s)\frac{d(\xi_i,\xi_j)}{dM_{ij}}(s,t)\right\}\frac{d(\xi_j,\xi_i)}{dM_{ji}}(t,u)M_j(du)\right\}\xi_i(du),\ldots.$$

Proof. For simplicity let $i=1$ and $j=2$. By 2.3,

(1) $$\frac{d(x,\xi_1)}{dM_1} \in L_2(R,\mathscr{B},M_1) \quad\text{and}\quad P_1x = \int_R \frac{d(x,\xi_1)}{dM_1}(s)\xi_1(ds).$$

Replacing x by P_1x and ξ_1 by ξ_2 in (1) we get

(2) $$\frac{d(P_1x,\xi_2)}{dM_2} \in L_2(R,\mathscr{B},M_2) \quad\text{and}\quad P_2P_1x = \int_R \frac{d(P_1x,\xi_2)}{dM_2}(t)\xi_2(dt).$$

Let $B\in\mathscr{B}_2$. Then we have

$$(P_1x,\xi_2(B)) = \left(\int_R \frac{d(x,\xi_1)}{dM_1}(s)\xi_1(ds),\ \xi_2(B)\right) \qquad \text{by (1)}$$
$$= \int_R \frac{d(x,\xi_1)}{dM_1}(s)(\xi_1(ds),\xi_2(B)) \qquad \text{by 2.4}$$

(3) $$= \int_R \frac{d(x,\xi_1)}{dM_1}(s)\frac{d(\xi_1,\xi_2(B))}{dM_1}(s)M_1(ds) \qquad \text{by 3.8(a)}$$
$$= \int_R \frac{d(x,\xi_1)}{dM_1}(s)\left\{\int_B \frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)M_2(dt)\right\}M_1(ds) \qquad \text{by 3.8(b)}$$
$$= \int_B\left\{\int_R \frac{d(x,\xi_1)}{dM_1}(s)\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)M_1(ds)\right\}M_2(dt),$$

where in the last step, the change of order of integration can be justified as follows:

By the Schwarz inequality, we have

$$\int_R \left|\frac{d(x,\xi_1)}{dM_1}(s)\right| \left|\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)\right| M_1(ds)$$

$$\leqq \left[\int_R \left|\frac{d(x,\xi_1)}{dM_1}(s)\right|^2 M_1(ds)\right]^{1/2} \times \left[\int_R \left|\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)\right|^2 M_1(ds)\right]^{1/2} \leqq CK,$$

where by (1), $C=[\int_R |(d(x,\xi_1)/dM_1)(s)|^2 M_1(ds)]^{1/2}<\infty$, and by our hypothesis and Assumption 3.5, $K=M_2-\text{ess. lub.}_{t\in R} |(d(\xi_1,\xi_2)/dM_{12})(\cdot,t)|_{2,M_1}<\infty$. Then

$$\int_B \left\{\int_R \left|\frac{d(x,\xi_1)}{dM_1}(s)\right| \left|\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)\right| M_2(dt)\right\} \leqq CKM_2(B) < \infty.$$

Hence Fubini's Theorem (cf. [3, p. 148]) can be applied.

Since M_2 is σ-finite, there exist countably many disjoint sets N_i such that $\sigma_{M_2}=\bigcup N_i$, where σ_{M_2} is the spectrum of M_2 (cf. 1.3(iii)), and $M_2(N_i)<\infty$. If $t\in N_i$, taking in (3) $B=N_i\cap\Delta_{t,h}$, where $\Delta_{t,h}=\{s: t-h<s<t+h\}$ we get

$$(P_1x, \xi_2(N_i\cap\Delta_{t,h})) = \int_{N_i\cap\Delta_{t,h}} \left\{\int_R \frac{d(x,\xi_1)}{dM_1}(s)\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,u)M_1(ds)\right\}M_2(du).$$

Hence by the Besicovitch Theorem 1.3

$$\text{(4)} \qquad \begin{aligned} \frac{d(P_1x,\xi_2)}{dM_2}(t) &= \lim \frac{(P_1x, \xi_2(N_i\cap\Delta_{t,h}))}{M_2(N_i\cap\Delta_{t,h})}, \quad \text{as } h\to 0 \\ &= \int_R \frac{d(x,\xi_1)}{dM_1}(s)\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)M_1(ds) \quad \text{a.e. } t\in N_i(M_2). \end{aligned}$$

Since $\{N_i\}$ is a countable collection, and $M_2(R-\sigma_{M_2})=0$, (4) holds a.e. $t(M_2)$. By (2), we see that the function on the right-hand side of (4) is in $L_2(R,\mathscr{B},M_2)$ and that

$$P_2P_1x = \int_R \left\{\int_R \frac{d(x,\xi_1)}{dM_1}(s)\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)M_1(ds)\right\}\xi_2(dt).$$

This completes the proof of the first relation in (a) and the first two relations in (b).

To obtain the expression for P_2P_1x given by (b), we essentially made use of the fact that

$$M_2-\underset{t\in R}{\text{ess. lub.}}\left|\frac{d(\xi_1,\xi_2)}{dM_{12}}(\cdot,t)\right|_{2,M_1} < \infty.$$

$$\frac{d(x,\xi_1)}{dM_1}(\cdot)\in L_2(R,\mathscr{B},M_1),$$

$$P_1x = \int_R \frac{d(x,\xi_1)}{dM_1}\,d\xi_1.$$

Now since

$$M_1-\underset{t\in R}{\text{ess. lub.}}\left|\frac{d(\xi_2,\xi_1)}{dM_{21}}(\cdot,u)\right|_{2,M_2}<\infty,$$

$$\phi(\cdot) = \int_R \frac{d(x,\xi_1)}{dM_1}(s)\frac{d(\xi_1,\xi_2)}{dM_{12}}(s;\,\cdot)M_1(ds)\in L_2(R,\mathscr{B},M_2),$$

$$P_2P_1x = \int_R \phi\, d\xi_2,$$

therefore in exactly the same way as before we can obtain the expression for $P_1P_2P_1x$ given in (b). The rest of the proof may be completed by induction. (Q.E.D.)

The following corollary is an immediate consequence of this theorem and 1.2.

COROLLARY 3.10. *With the notation of* 3.1, *if* ξ_i $(i=1,2)$ *are o.s.* $\mathscr{H}$*-valued measures satisfying Assumption* 3.5 *and if* P_i $(i=1,2)$ *and* P *are the projection operators on* $\mathscr{H}$ *onto the cyclic subspaces* $\mathscr{M}_i=\mathfrak{S}\{\xi_i(B),B\in\mathscr{B}_i\}$ $(i=1,2)$ *and* $\mathfrak{S}\{\mathscr{M}_1+\mathscr{M}_2\}$, *then for each* $x\in\mathscr{H}$,

$$\begin{aligned}
Px = {} & \int_R \frac{d(x,\xi_1)}{dM_1}(s)\xi_1(ds)+\int_R \frac{d(x,\xi_2)}{dM_2}(s)\xi_2(ds)\\
&-\int_R\left\{\int_R \frac{d(x,\xi_2)}{dM_2}(s)\frac{d(\xi_2,\xi_1)}{dM_{21}}(s,t)M_2(ds)\right\}\xi_1(dt)\\
&-\int_R\left\{\int_R \frac{d(x,\xi_1)}{dM_1}(s)\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)M_1(ds)\right\}\xi_2(dt)\\
&+\int_R\left\{\int_R\left\{\int_R \frac{d(x,\xi_1)}{dM_1}(s)\frac{d(\xi_1,\xi_2)}{dM_{12}}(s,t)M_1(ds)\right\}\frac{d(\xi_2,\xi_1)}{dM_{21}}(t,u)M_2(dt)\right\}\xi_1(du)\\
&+\int_R\left\{\int_R\left\{\int_R \frac{d(x,\xi_1)}{dM_2}(s)\frac{d(\xi_2,\xi_1)}{dM_{21}}(s,t)M_2(ds)\right\}\frac{d(\xi_1,\xi_2)}{dM_{12}}(t,u)M_1(dt)\right\}\xi_2(du)\\
&-\cdots.
\end{aligned}$$

4. **Application of the alternating projections theorem to bivariate stationary stochastic processes.** Let $\mathbf{x}(t)=(x_i(t))_{i=1}^q\in\mathscr{H}^q$ be a q-variate, stationary SP with the shift group $(U_t,t\in R)$. Here $\mathscr{H}^q$ is the cartesian product of a complex Hilbert space $\mathscr{H}$ with itself q-times, and $(U_t,t\in R)$ is a strongly continuous group of unitary operators on $\mathscr{H}$ such that $x_i(t+s)=U_tx_i(s)$ for all s, $t\in R$ $(1\leqq i\leqq q)$. By Stone's Theorem (cf. [8, p. 383]), there exists a spectral measure E on the family $\mathscr{B}$ of Borel subsets of the real line R such that $U_t=\int_{-\infty}^{\infty}e^{-it\lambda}E(d\lambda)$. The $q\times q$ non-negative, hermitian matrix-valued function $\mathbf{F}(\lambda)=2\pi(\mathbf{E}(-\infty,\lambda]\mathbf{x}(0),\mathbf{x}(0))$ is called the spectral distribution of the SP. If $\mathbf{F}$ is absolutely continuous with respect to the Lebesgue measure (Leb), its derivative $\mathbf{F}'$ is called the *spectral density* of the SP.

Let $\mathcal{M}(t)$ be the past-present subspace of the SP $(\mathbf{x}(t), t \in R)$ up to time t in $\mathcal{H}^q$, i.e., $\mathcal{M}(t)=\mathfrak{S}\{\mathbf{x}(s), s \leqq t\}$. Let $\mathcal{M}(-\infty)$ be its remote past subspace in $\mathcal{H}^q$, i.e., $\mathcal{M}(-\infty)=\bigcap_{t\leqq 0} \mathcal{M}(t)$.

An important result is that if $\mathcal{M}(-\infty)=\{0\}$, then $\mathbf{F}$ is a.c. with respect to Leb, and $\mathbf{x}(t)=\int_{-\infty}^{t} \mathbf{C}(t-s)\boldsymbol{\xi}(ds)$, where $\mathbf{C}(\cdot)$ and $\mathbf{F}'(\cdot)$ are related by a matrix-valued function $\boldsymbol{\Phi}$ on R such that

$$\boldsymbol{\Phi}(\lambda)\boldsymbol{\Phi}^*(\lambda) = \mathbf{F}'(\lambda) \text{ a.e. } \lambda \quad \text{and} \quad \boldsymbol{\Phi}(\lambda) = \int_0^\infty \mathbf{C}(t)e^{i\lambda t}\,dt \in \mathbf{L}_2(R, \mathcal{B}, \text{Leb}),$$

and if $\boldsymbol{\Phi}^+$ denotes the holomorphic extension of $\boldsymbol{\Phi}$ to the upper half-plane, $\boldsymbol{\Phi}^+(i)$ is nonnegative hermitian, and

$$\det \boldsymbol{\Phi}^+(i) = \exp \frac{1}{2\pi}\int_{-\infty}^{\infty} \log \det \mathbf{F}'(\lambda)\,\frac{d\lambda}{1+\lambda^2}.$$

The function $\boldsymbol{\Phi}$ is called the *generating function* of the SP. It is not hard to show (cf. [5, p. 1372] and [11, p. 33]) that for each bounded interval $(a, b]$

$$\boldsymbol{\xi}(a, b] = 2^{-1/2}\left\{\mathbf{h}(b)-\mathbf{h}(a)+\int_a^b \mathbf{h}(s)\,ds\right\},$$

where $(\mathbf{h}(t), t \in R)$ is the weakly Markovian SP which is associated with the SP $(\mathbf{x}(t), t \in R)$.

We will need the following theorem, the proof of which is omitted and may be found in [11].

THEOREM 4.1. *Let* $(\mathbf{x}(t), t \in R)$ *be a q-variate, stationary* SP *with spectral distribution* $\mathbf{F}$. *Then* (a) $(\mathbf{x}(t), t \in R)$ *is purely nondeterministic, i.e.,* $\mathcal{M}(-\infty)=\{0\}$ *iff* $\mathbf{F}$ *is absolutely continuous with respect to the Lebesgue measure. If* (a) *holds, then*

(b) $\mathfrak{S}\{\boldsymbol{\xi}(a, b], -s<a\leqq b<t\}=\mathfrak{S}\{\mathbf{x}(u): s<u<t\}$.

(c) $\boldsymbol{\xi}$ *can be uniquely extended to the family* $\mathcal{B}$ *of Borel subsets of the real line R. If the* SP *is of rank q, or equivalently* $\log \det \mathbf{F}'(\lambda)/(1+\lambda^2) \in L_1(R, \mathcal{B}, \text{Leb})$, *then for each Borel set B of finite Lebesgue measure this extension has the representation*

$$\boldsymbol{\xi}(B) = (2\pi)^{1/2}\int_{-\infty}^{\infty} \hat{\chi}_B(\lambda)\boldsymbol{\Phi}^{-1}(\lambda)\mathbf{E}(d\lambda)\mathbf{x}(0),$$

where $\hat{\chi}_B(\lambda)=(2\pi)^{-1/2}\int_B e^{-i\lambda t}\,dt$.

(d) *The extension* $\boldsymbol{\xi}$ *is an o.s.* $\mathcal{H}^q$*-valued measure over* $(R, \mathcal{B})$, *whose associated spectral measure is* $\text{Leb}(\cdot)\mathbf{I}_q$, *where* I_q *is the identity matrix of order q.*

DEFINITION 4.2. $\boldsymbol{\xi}$ as defined in 4.1(c) is called the *canonical measure* of the SP $(\mathbf{x}(t), t \in R)$.

Now let $(\mathbf{x}(t), t \in R)$ be a bivariate, purely nondeterministic SP. Let $\mathbf{T}$ be the projection operator on $\mathcal{H}^2$ onto the subspace $\mathcal{M}(0)$, and P be the projection operator on $\mathcal{H}$ onto the subspace $\mathfrak{S}\{\mathcal{M}_1(0)+\mathcal{M}_2(0)\}$, where $\mathcal{M}_1(0)$ and $\mathcal{M}_2(0)$ are the past-present subspaces of the component processes $x_1(t)$ and $x_2(t)$. It is easy to see (cf. [12, I, p. 131]) that the components of $\mathbf{Tx}(t)$ are precisely $Px_1(t)$ and

$Px_2(t)$. Applying the results of §3 we will be able to obtain convergent infinite series expressions for $Px_1(t)$ and $Px_2(t)$, and hence the best linear prediction $\mathbf{T}\mathbf{x}(t)$ is determinable.

Since $(\mathbf{x}(t), t \in R)$ is purely nondeterministic, so are its component processes $(x_1(t), t \in R)$ and $(x_2(t), t \in R)$ (cf. [11, p. 79]). Accordingly ξ_1 and ξ_2 will denote the canonical measures of the processes $(x_1(t), t \in R)$ and $(x_2(t), t \in R)$. Also $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$ will denote the past-present subspaces of $(x_1(t), t \in R)$ and $(x_2(t), t \in R)$ up to time t in $\mathcal{H}$.

We will be able to effect our algorithm under the following assumption.

Assumption 4.3. The spectral density $\mathbf{F}' = [F'_{ij}]$, $1 \leqq i, j \leqq 2$, of the bivariate, purely nondeterministic SP $(\mathbf{x}(t), t \in R)$ satisfies the condition

$$F'_{12}/(F'_{11})^{1/2}(F'_{22})^{1/2} \in L_1(R, \mathcal{B}, \text{Leb})(^4).$$

Remark 4.4. We assert that the SP has rank 2. For if its rank were less than 2, then $F'_{11}F'_{22} - F'_{12}F'_{21} = \det F' = 0$ a.e. on R (cf. [11, p. 36]). Hence $F'_{12}/(F'_{11})^{1/2}(F'_{22})^{1/2} = 1 \notin L_1(R, \mathcal{B}, \text{Leb})$ which contradicts Assumption 4.3.

The component processes $(x_1(t), t \in R)$ and $(x_2(t), t \in R)$ are purely nondeterministic, and hence have the generating function ϕ_1, ϕ_2. We also note that since $\det F' \geqq 0$, $F'/(F'_{11})^{1/2}(F'_{22})^{1/2} \leqq 1$. Hence by Assumption 4.3,

$$F'_{12}/(F'_{11})^{1/2}(F'_{22})^{1/2} \in L_\delta(R, \mathcal{B}, \text{Leb}), \quad 1 \leqq \delta \leqq \infty.$$

Theorem 4.5. *Let* (i) *the* SP $(\mathbf{x}(t), t \in R)$ *satisfy Assumption* 4.3.

(ii) ξ_i $(i=1, 2)$ *be the canonical measure of the component process* $(x_i(t), t \in R)$ $(i=1, 2)$. *Then Assumption* 3.5 *is satisfied, i.e., there exist Borel measurable functions* φ_{12} *and* φ_{21} *on* R^2 *such that*

$$\text{(1)} \qquad \begin{aligned} &\varphi_{ij}(\cdot, t) \in L_2(R, \mathcal{B}, \text{Leb}) \textit{ a.e. } t, \\ &\operatorname*{ess.\,lub.}_{t \in R} |\varphi_{ij}(\cdot, t)|_{2,\text{Leb}} < \infty. \end{aligned}$$

(2) *For each* $B \in \mathcal{B}_{ij}$, $\chi_B \varphi_{ij} \in L_1(R^2, \mathcal{B} \times \mathcal{B}, \text{Leb} \times \text{Leb})$.

(3) *For each* $B \times C \in \mathcal{B}_i \times \mathcal{B}_j$,

$$(\xi_i(B), \xi_j(C)) = \iint_{B \times C} \varphi_{ij}(s, t)\, ds\, dt.$$

In fact if

$$\gamma_{ij}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\lambda} \frac{F'_{ij}(\lambda)}{\phi_i(\lambda)\bar{\phi}_j(\lambda)}\, d\lambda(^5),$$

then

$$\varphi_{ij}(s, t) = \gamma_{ij}(s-t),$$

where ϕ_i $(i=1, 2)$ *is the generating function of the component process* $(x_i(t), t \in R)$ $(i=1, 2)$.

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(4) By Cramér's Theorem (cf. [2, p. 221]) it is easy to see that this assumption is not void.

(5) Since $|\phi_i|^2 = F'_{ii}$ $(i=1, 2)$, by 4.4, $F'_{ij}/\phi_i\bar{\phi}_j$ is in $L_\delta(R, \mathcal{B}, \text{Leb})$, $1 \leqq \delta \leqq \infty$.

Proof. Let $\gamma_{ij}(t)=(1/2\pi)\int_{-\infty}^{\infty} e^{-it\lambda}\,(F'_{ij}(\lambda)/\phi_i(\lambda)\bar{\phi}_j(\lambda))\,d\lambda$. We define φ_{ij} on R^2 by

$$\varphi_{ij}(s,t) = \gamma_{ij}(s-t).$$

Since $|\phi_i|^2=F'_{ii}$, by (i) and 4.4, $F'_{ij}/\phi_i\bar{\phi}_j \in L_2(R, \mathscr{B}, \text{Leb})$. Hence by Plancherel's Theorem we have $\int_{-\infty}^{\infty} |\varphi_{ij}(s,t)|^2\,ds=\int_{-\infty}^{\infty} |\gamma_{ij}(s-t)|^2\,ds=(1/2\pi)|F'_{ij}/\phi_i\bar{\phi}_j|^2_{2,\text{Leb}}<\infty$, so that ess. $\text{lub.}_{t\in R}\,|\varphi_{ij}(\cdot,t)|_{2,\text{Leb}}=(1/2\pi)|F'_{ij}/\phi_i\bar{\phi}_j|_{2,\text{Leb}}<\infty$. Hence (1) is proved.

Also by (i) and 4.4, $\varphi_{ij}\in L_\infty(R^2, \mathscr{B}\times\mathscr{B}, \text{Leb}\times\text{Leb})$ and hence (2) is satisfied.

By 4.1, for each $B\in\mathscr{B}_i$ $(i=1,2)$ we have

$$\xi_i(B) = (2\pi)^{1/2}\int_{-\infty}^{\infty} \hat{\chi}_B(\lambda)\phi_i^{-1}(\lambda)E(d\lambda)x(0) \quad (i = 1, 2).$$

From (i) it follows that $F'_{ij}/\phi_i\bar{\phi}_j\in L_1(R, \mathscr{B}, \text{Leb})$, so that by Fubini's Theorem for each $B\times C\in\mathscr{B}_i\times\mathscr{B}_j$,

$$\begin{aligned}(\xi_i(B), \xi_j(C)) &= 2\pi\left(\int_{-\infty}^{\infty} \hat{\chi}_B(\lambda)\phi_i^{-1}(\lambda)E(d\lambda)x_i(0), \int_{-\infty}^{\infty} \hat{\chi}_C(\lambda)\phi_j^{-1}(\lambda)E(d\lambda)x_i(0)\right)\\
&= 2\pi\int_{-\infty}^{\infty} \hat{\chi}_B(\lambda)\overline{\hat{\chi}_C(\lambda)}\phi_i^{-1}(\lambda)\overline{\phi_j^{-1}(\lambda)}(E(d\lambda)x_i(0), x_j(0))\\
&= \int_{-\infty}^{\infty} \hat{\chi}_B(\lambda)\overline{\hat{\chi}_C(\lambda)}\cdot\frac{F'_{ij}(\lambda)}{\phi_i(\lambda)\bar{\phi}_j(\lambda)}\,d\lambda \qquad \text{(cf. [11, p. 82])}\\
&= \frac{1}{2\pi}\iint_{B\times C}\left\{\int_{-\infty}^{\infty} e^{-i(s-t)\lambda}\,\frac{F'_{ij}(\lambda)}{\phi_i(\lambda)\bar{\phi}_j(\lambda)}\,d\lambda\right\}ds\,dt\\
&= \iint_{B\times C}\gamma_{ij}(s,t)\,ds\,dt. \qquad \text{(Q.E.D.)}\end{aligned}$$

We now state the main theorem of this section.

MAIN THEOREM 4.6. *Let* (i) *the bivariate* SP $(\mathbf{x}(t), t\in R)$ *satisfy Assumption* 4.3.

(ii) $\mathscr{M}_i(0)$ $(i=1,2)$ *be the past-present subspace of the component process* $(x_i(t), t\in R)$ $(i=1,2)$, *and let* P_1, P_2, *and* P *be the projection operators on* $\mathscr{H}$(6) *onto the subspaces* $\mathscr{M}_1(0)$, $\mathscr{M}_2(0)$, *and* $\mathfrak{S}\{\mathscr{M}_1(0)+\mathscr{M}_2(0)\}$.

(iii) $\phi_i(\lambda)=\int_0^\infty e^{i\lambda t}c_i(t)\,dt$,

$$\gamma_{ij}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\lambda}\,\frac{F'_{ij}(\lambda)}{\phi_i(\lambda)\bar{\phi}_j(\lambda)}\,ds,$$

where ϕ_i $(i=1,2)$ *is the generating function of the component process* $(x_i(t), t\in R)$, $1\leq i\leq 2$.

(iv) ξ_i $(i=1,2)$ *be the canonical measure of the component process* $(x_i(t), t\in R)$ $(i=1,2)$.

(6) $\mathscr{H}$ is a Hilbert space such that $x_i(t)\in\mathscr{H}$ for all $t\in R$, $i=1,2$.

Then for each $\tau \geqq 0$,

$$Px_i(\tau) = \int_0^\infty c_i(\tau+s)\xi_i(-ds) + \int_0^\infty \left\{\int_{-\tau}^0 c_i(\tau+s)\gamma_{ij}(t-s)ds\right\}\xi_j(-ds)$$
$$-\int_0^\infty \left\{\int_0^\infty \left\{\int_{-\tau}^0 c_i(\tau+s)\gamma_{ij}(t-s)\,ds\right\}\gamma_{ji}(u-t)\,dt\right\}\xi_i(-du)+\cdots.$$

Proof. By the third paragraph of §4 and 4.1(b),

$$x_i(\tau) = \int_{-\tau}^\infty c_i(\tau+s)\xi_i(-ds), \tag{1}$$
$$\mathcal{M}_i(0) = \mathfrak{S}\{\xi_i(B),\ B \subseteq (-\infty, 0] \text{ and } B \in \mathcal{B}_i\}.$$

By 4.5(c), $(d(\xi_i, \xi_j)/dM_{ij})(s, t) = \gamma_{ij}(s-t)$ and Assumption 3.5 is satisfied. Hence by (i) and 3.9, we have

$$P_ix_i(\tau) = \int_0^\infty c_i(\tau+s)\xi_i(-ds),$$
$$P_jx_i(\tau) = \int_0^\infty \left\{\int_{-\tau}^\infty c_i(\tau+s)\gamma_{ij}(t-s)\,ds\right\}\xi_j(-dt),$$
$$P_iP_jx_i(\tau) = \int_0^\infty \left\{\int_0^\infty \left\{\int_{-\tau}^\infty c_i(\tau+s)\gamma_{ij}(t-s)\,ds\right\}\gamma_{ji}(u-t)\right\}\xi_i(-du),$$
$$P_jP_ix_j(\tau) = \int_0^\infty \left\{\int_0^\infty c_i(\tau+s)\gamma_{ij}(t-s)\,ds\right\}\xi_j(-dt).$$

Hence, after some simplification, we have the result by 3.10. (Q.E.D.)

The following is an immediate corollary to this theorem.

COROLLARY 4.7. *With the notation of* 4.6, *if* Q *is the projection operator on* $\mathcal{H}$ *onto the orthogonal complement of* $\mathfrak{S}\{\mathcal{M}_1(0)+\mathcal{M}_2(0)\}$ *in* $\mathcal{H}$ *then for each* $\tau \geqq 0$

$$Qx_i(\tau) = \int_{-\tau}^0 c_i(\tau+s)\xi_i(ds) - \int_0^\infty \left\{\int_{-\tau}^0 c_i(\tau+s)\gamma_{ij}(t-s)\,ds\right\}\xi_j(-dt)$$
$$+\int_0^\infty \left\{\int_0^\infty \left\{\int_{-\tau}^0 c_i(t+s)\gamma_{ij}(t-s)\,ds\right\}\gamma_{ji}(u-t)\,dt\right\}\xi_i(-du)-\cdots.$$

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